Over 400000 Euler i.c.s with $T^* < 200$ and their distribution in the initial velocities' plane (supplementary material to [1])

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An additional numerical search for periodic orbits using the efficient approach from [1] is conducted. The same quadratic search grid with step-size 3.2^{-14} as in [1] is used. The scale-invariant periods T^* are now longer - $T^* < 200$. 421562 i.c.s corresponding to 210038 periodic solutions of type (II) (presented by two different i.c.s.) and 1486 solutions of type (I) (presented by one i.c.) are found. The found i.c.s are computed with Newton's method up to convergence with residual less than 10^{-60}. They are shown with small black dots in Figure 1. The i.c.s from paper [2] for periods $T^* < 200$ are presented by red dots. These solutions are rediscovered and therefore there is one small black dot on each red dot. The i.c.s from [2] for $T^* > 200$ are in blue. The stability regions computed on a relatively coarse grid with step-size 2^{-9} are in yellow. The stability regions' points (in yellow) are defined as in [3], i.e. they are the points for which the scale-invariant escape time T^*_{esc} is greater than some long upper time limit, meaning that this time limit is insufficient for escape. Here we choose the upper time limit to be 2000 (not too large number), i.e. $T^*_{esc} > 2000$ for stable points. The escape criterion is as in [3], namely the maximum distance between bodies $r_{max} > 5d$, where $d = 3/|E|$ is the average triple system size.

Note however, that the yellow regions have to be regarded qualitatively, as (1) a finer grid should be considered, (2) the results depend on the considered upper limit of the time and the escape criterion, and (3) a further accuracy verification is also needed. It is also important to remark that the results for the stability regions qualitatively match those in [3].

As revealed in [2], due to the sensitivity of the solutions on the initial conditions, we need to use a high-order method applied with high precision in order to follow the trajectories for long enough time. So, using high precision is a crucial decision for searching periodic orbits. However, we need more than using high precision for an efficient implementation of the search-grid method in combination with the Newton's method. The efficiency depends also on the nonlinear equation we solve, as shown in [1].

Now we will give a simple Lyapunov exponents' argument, which tries to explain the observed high efficiency of the "half period" approach in [1] and the obtained huge number of periodic i.c.s. Let us assume unstable periodic orbits and that for small separation $d(t) = \|p(t) - \widetilde{p}(t)\|_2$ between adjacent trajectories the exponential law of divergence is satisfied:

$$d(t) \approx d(0)e^{\lambda t}, \quad \lambda - \text{the Lyapunov exponent}$$

Here $p(t)$ is a given unstable periodic orbit and $\widetilde{p}(t)$ is an approximation of it. What is the effect of the integration time division by two, i.e. the effect of solving the Euler equation at $t = T/2$ instead of solving the periodic equation at $t = T$? The benefit is not simply reducing the computational time by two, but much more. We have a "square root effect" on the distance between adjacent trajectories, meaning that:

$$d(T/2) \approx d(0)\sqrt{e^{\lambda T}}, \quad T - \text{the period}$$

Let us take for example $d(0) \approx 10^{-4}$ and take $e^{\lambda T} \approx 10^8$. Then $d(T/2) \approx 1$, but $d(T) \approx 10^4$. In the first case we can expect convergence of Newton's method. In the second case the above approximate exponential low is in fact not valid, the trajectories are already so "mixed" at $t = T$ for $d(0) \approx 10^{-4}$, that Newton's method does not work (regardless that we have computed the trajectories exactly). Note that to obtain a similar result for the example case above by solving the standard periodic equation, we have to take $d(0) \approx 10^{-8}$, which means to use so fine search grid, that the high precision computations are so time consuming that they are practically impossible. Of course, each captured solution by solving the Euler equation is tested after that for convergence with Newton's method for the standard periodicity equation, but now the initial conditions are already accurate enough and the convergence test always passes.

[1] Hristov, Hristova. "An efficient approach for searching three-body periodic orbits passing through Eulerian configuration". Astronomy and Computing, 49, (2024): 100880

[2] Li, and Liao. "More than six hundred new families of Newtonian periodic planar collisionless three-body orbits." Science China Physics, Mechanics & Astronomy 60 (2017): 1-7.

[3] Martynova, Orlov, and Rubinov. "The structure of non-hierarchical triple system stability regions." Astronomy reports 53 (2009): 710-721.

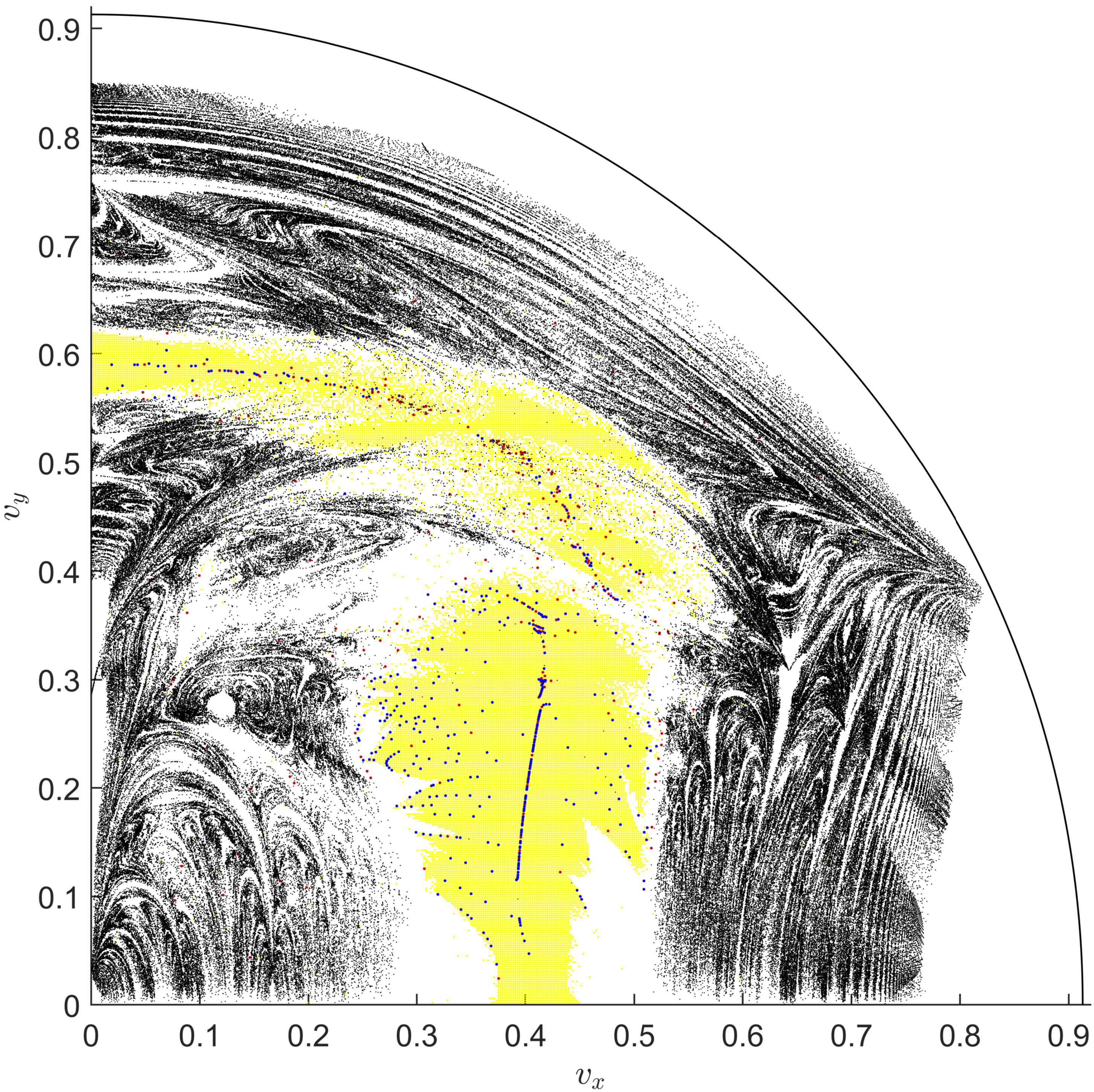

Figure 1 (Distribution of i.c.s in the initial velocities' plane. Stability regions as defined in [3] are in yellow.)