Linear stability computations

Let $X(t; X^0), 0 \leq t \leq T$, be a periodic solution with period T and initial condition $X(0) = X^0$. The stability of $X(t; X^0)$ is manifested by the way of behaviour of the neighboring trajectories. A trajectory that starts from the perturbed initial vector $X^0 + \delta X^0$ will be displaced after one period T by:

$$\delta X(T; X^0) = X(T; X^0 + \delta X^0) - X(T; X^0)$$

Expanding this equation in a multivariable linear approximation we have:

$$\delta X(T; X^0) \approx M[T; X^0]\delta X^0$$

where $M_{ij}[T; X^0] = \frac{\partial X_i(T; X^0)}{\partial {X_j}^0}$ are the elements of the so called monodromy matrix $M[T; X^0]$. The linear stability of the given periodic solution $X(t; X^0)$ is determined by the eigenvalues λ of the 12×12 in our case monodromy matrix $M(T; X^0)$ [1].

The elements M_{ij} of the monodromy matrix M are computed in the same way as the partial derivatives in the linear systems for Newton's method - with the multiple precision Taylor Series Method using the rules of automatic differentiation (see for details [2]). The initial conditions $M(0) = I$, where I is the identity matrix have to be set.

The eigenvalues of M come in pairs or quadruplets: $(\lambda, \lambda^{-1}, \lambda^*, \lambda^{*-1})$. They are of four types:

1) Elliptically stable - $\lambda = \exp(\pm 2\pi i\nu)$, where $\nu > 0$ (real) is the stability angle. In this case the eigenvalues are on the unit circle. Angle ν describes the stable revolution of adjacent trajectories around a periodic orbit.

2) Marginally stable - $\lambda = \pm 1$.

3) Hyperbolic - $\lambda = \pm\exp(\pm\mu)$, where $\mu > 0$ (real) is the Lyapunov exponent.

4) Loxodromic - $\lambda = \exp(\pm\mu \pm i\nu)$, μ, ν (real).

Eight of the eigenvalues of M are equal to 1 [1]. The other four determine the linear stability. Here we are interested in elliptically stable orbits, i.e. the four eigenvalues to be $\lambda_j = \exp(\pm 2\pi i \nu_j), \nu_j > 0, j = 1, 2$. For computing the eigenvalues we use a Multiprecision Computing Toolbox [3] for MATLAB® [4]. First the elements of M are obtained with 130 correct digits and then two computations with 70 and 130 digits of precision are made with the toolbox. The four eigenvalues under consideration are verified by a check for matching their first digits obtained by the two computations with 70 and 130 digits of precision. The first digits of the corresponding condition numbers are also checked. More than 25 of the first digits for the eigenvalues and for the condition numbers are the same. The four eigenvalues and their absolute values for all trivial and all nontrivial choreographies are given with 25 correct digits in two separate files.

References

[1] G. Roberts, "Linear stability analysis of the figure-eight orbit in the three-body problem." Ergodic Theory and Dynamical Systems 27.6 (2007): 1947-1963.

[2] I. Hristov, R. Hristova, I. Puzynin, T. Puzynina, Z. Sharipov, Z. Tukhliev, "Newton's method for computing periodic orbits of the planar three-body problem." arXiv preprint arXiv:2111.10839 (2021).

[3] Advanpix LLC., Multiprecision Computing Toolbox for MATLAB, `http://www.advanpix.com/`, Version 4.9.0 Build 14753, 2022-08-09

[4] MATLAB version 9.12.0.1884302 (R2022a), The Mathworks, Inc., Natick, Massachusetts, 2021